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STABILIZATION OF WEAKLY LINEAR SYSTEMS*

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The problem of stabilizing bilinear systems, characterized by the presence of a small parameter in the bilinear part of the system, is considered. The result is an approximate method for synthesizing a stabilizing control /1-3/ in bilinear systems, in the case of a performance index. Estimates are derived for the error with respect to the performance index.

1. *Statement of the problem.* Suppose we are given a bilinear control system

$$x^{\cdot} = \varepsilon N(t)xu + B(t)u;\ x \in R_n;\quad x(0) = x_0;\ t \geqslant 0 \tag{1.1}$$

Here $N(t)$ is a measurable and bounded $n \times n$ matrix for $t \geqslant 0$; $B(t) \in R_n$ is a vector-valued function, also measurable and bounded for $t \geqslant 0$. The problem is to determine a scalar control in the class U of bounded controls $u = u(t, x)$, $\varepsilon \geqslant 0$ is a small parameter.

We wish to synthesize an optimal control in class U, which stabilizes system (1.1). The performance index is

$$J(u) = \int (x'Q(t)x + \lambda(t)^{-1}u^2)\,dt \tag{1.2}$$

Here $Q(t)$ is a continuous, bounded, uniformly positive definite $n \times n$ matrix, and $\lambda(t)$ is a positive definite scalar function; the prime denotes transposition. Integration with respect to t is always from 0 to ∞.

2. *Successive approximations algorithm.* Let us assume that for the values of ε under consideration problem (1.1), (1.2) has a solution. Bellman's equation is

$$\inf_{u \in U} [\partial V/\partial t + u(B(t) + \varepsilon N(t)x)'\partial V/\partial x + x'Q(t)x + \lambda(t)^{-1}u^2] = 0, \quad (V = V(t, x)) \tag{2.1}$$

It follows from (2.1) that the following expression defines an optimal control:

$$u_*(t, x) = -\frac{1}{2}\lambda(t)(B(t) + \varepsilon N(t)x)'\,\partial V/\partial x \tag{2.2}$$

Expand the function V in powers of ε:

$$V = V_0(t, x) + \varepsilon V_1(t, x) + \ldots \tag{2.3}$$

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To determine $V_i(t, x)$ we must substitute (2.2) into (2.1), and then substitute (2.3) into the result and equate the coefficients of like powers of ε to zero. This gives the following linear equations for $V_i(t, x)$:

$$\frac{\partial V_i}{\partial t} - \frac{1}{4}\Big[\sum_{k=0}^{i} \frac{\partial V_k'}{\partial x} B_1 \frac{\partial V_{i-k}}{\partial x} + \sum_{k=0}^{i-1} \frac{\partial V_k'}{\partial x} B_2 \frac{\partial V_{i-k-1}}{\partial x} + \sum_{k=0}^{i-2} \frac{\partial V_k'}{\partial x} B_3 \frac{\partial V_{i-k-2}}{\partial x}\Big] = 0, \quad i \geqslant 1 \tag{2.4}$$

$$B_1 = B(t)\lambda(t)B'(t), \quad B_2 = B(t)\lambda(t)x'N'(t) + N(t)x\lambda(t)B'(t),$$
$$B_3 = N(t)x\lambda(t)x'N'(t)$$

When $i = 1$ the last sum vanishes. Eqs.(2.4) are solved in the class of continuously differentiable bounded functions. The Bellman function of the zeroth approximation is $V_0(t, x) = x'P(t)x$, where $P(t)$ is a continuous, bounded, positive definite $n \times n$ matrix. Under certain conditions $P(t)$ is the unique positive definite solution of the Riccati equation /4, 5/

$$P^{\cdot}(t) - P'(t)B_1(t)P(t) = -Q(t) \tag{2.5}$$

Thus, provided that Bellman's equation is solvable when $\varepsilon = 0$, the zeroth-approximation control is given by

$$u_0(t, x) = -\lambda(t)B'(t)P(t)x \tag{2.6}$$

When $i \geqslant 1$ the solution of Eq.(2.4) is given by

$$V_i(t, x) = -\int_t^{\infty} \frac{1}{4}\Big[\sum_{k=1}^{i-1} \frac{\partial V_k'(\tau, x, (\tau))}{\partial x} B_1(\tau) \frac{\partial V_{i-k}(\tau, x(\tau))}{\partial x} + \sum_{k=0}^{i-1} \frac{\partial V_k'(\tau, x(\tau))}{\partial x} B_2(\tau, x(\tau)) \frac{\partial V_{i-k-1}(\tau, x(\tau))}{\partial x} + \sum_{k=0}^{i-2} \frac{\partial V_k'(\tau, x(\tau))}{\partial x} B_3(\tau, x(\tau)) \frac{\partial V_{i-k-2}(\tau, x(\tau))}{\partial x}\Big] d\tau \tag{2.7}$$

$V_0(t, x) = x'P(t)x$.

Here $x(\tau)$ is the solution of system (1.1) for $\varepsilon = 0, \tau \geqslant t$, where the control is $u_0(\tau, x(\tau)) = -\lambda(\tau)B'(\tau)P(\tau)x(\tau)$ and the initial condition $x(t) = x$.

3. ***Estimation of the zeroth approximation.*** Let problem (1.1), (1.2) have a solution for some given ε and for $\varepsilon = 0$. We wish to estimate the difference $J(u_0) - J(u_*)$. If this difference is of the order of ε, formula (2.6) yields a zeroth approximation to the optimal control $u_*(t, x)$ in problem (1.1), (1.2). Assume that the following inequality holds (the letter C will denote various positive constants):

$$x'Q(t)x - \varepsilon x'P'(t)B'(t)\lambda(t)x'N'(t)P(t)x \geqslant C|x|^2 \tag{3.1}$$

Then there exists a zeroth-approximation performance index $J_0(u)$, which differs from $J(u)$ by a quantity of the order of ε:

$$F(t, x, u) = x'Q(t)x + \lambda(t)^{-1}u^2 - \varepsilon ux'N'(t)\partial V_0/\partial x \geqslant C|x|^2 \tag{3.2}$$

$$J_0(u) = \int F(t, x, u)\, dt \tag{3.3}$$

Since by condition (3.2) the integrand $F(t, x, u)$ is positive definite as a function of x, the control $u_0(t, x)$ is optimal for (1.1) in the sense of (3.3). Hence $V_0(t, x)$ is the Bellman function. Consequently,

$$\int F(t, x(t, u_0), u_0(t, x(t, u_0)))\, dt = V_0(0, x_0) \tag{3.4}$$

Here and below $x(t, u_0)$ is the trajectory of the system when the control $u_0(t, x)$ is applied. It follows from (3.2) that

$$\int F(t, x(t, u_0), u_0(t, x(t, u_0)))\, dt \geqslant C\int |x(t, u_0)|^2\, dt \tag{3.5}$$

Using (3.4) and (3.5), we obtain an estimate for solutions of (1.1) with $u = u_0(t, x)$:

$$\int |x(t, u_0)|^2 dt \leqslant CV_0(0, x_0) \tag{3.6}$$

Therefore, we can state that $J_0(u) < \infty$ and the control $u_0(t, x)$ is admissible for problem (1.1), (1.2).

We have

$$V(t, x) \leqslant V_0(t, x) + |J(u_0) - J_0(u_0)| = J_0(u_0) + \delta$$

or

$$J(u_*) \leqslant J_0(u_0) + \delta \tag{3.7}$$

Substituting (3.3) into (3.7) and estimating $J(u)$ for the control $u_0(t, x)$, we obtain

$$\delta \leqslant 2\varepsilon \int x'(t, u_0) P'(t) B(t) \lambda(t) x'(t, u_0) N'(t) P(t) x(t, u_0) dt$$

Consequently, we have the following upper bound for the error in the performance index:

$$J(u_*) \leqslant J_0(u_0) + \varepsilon C$$

A lower bound is established in analogous fashion. The final result is

$$0 \leqslant J(u_0) - J(u_*) \leqslant C\varepsilon \tag{3.8}$$

We assert that the control (2.6) makes system (1.1) asymptotically stable. the zeroth-approximation closed-loop system is asymptotically stable. Condition (3.6) makes it possible to use the first-approximation stability theorem of /6/. The result is that system (1.1), which is a closed-loop system relative to the zeroth-approximation control, is asymptotically stable.

4. Estimation of higher approximations. The i-th approximation control is determined by the formula

$$u_i(t, x) = u_0(t, x) - {}^1/_2 \lambda(t)(B(t) + \varepsilon N(t)x)' \partial W/\partial x$$
$$W = \sum_{k=1}^{i} \varepsilon^k V_k(t, x)$$

Here $u_0(t, x)$ is the zeroth-approximation control (2.6).

Suppose there exist functions $V_k(t, x)$ $(k \leqslant i)$ which are continuously differentiable with respect to both arguments and satisfy Eqs.(2.4). In addition, let us assume that

$$|W| \leqslant C|x|^2, \quad |\partial W/\partial x| \leqslant C|x|$$

Multiplying (2.4) by ε^i and summing over i, we get

$$\partial W/\partial t - \frac{1}{4}(\partial W'/\partial x)(B_1 + \varepsilon B_2 + \varepsilon^2 B_3)\partial W/\partial x = -x'Q(t)x + h_i \varepsilon^{i+1}$$

$$h_i = h_i(t, x, \varepsilon) = \frac{1}{2}\Big[\sum_{l=1}^{i}\sum_{j=0}^{l-1} \frac{\partial V_j'}{\partial x} B_1 \frac{\partial V_{i-l}}{\partial x}\varepsilon^j + \sum_{l=1}^{i-1}\sum_{j=0}^{l-1} \frac{\partial V_j'}{\partial x} B_2 \frac{\partial V_{i-j}}{\partial x}\varepsilon^l +$$
$$\sum_{l=1}^{i-2}\sum_{j=0}^{l-1} \frac{\partial V_j'}{\partial x} B_3 \frac{\partial V_{i-l}}{\partial x}\varepsilon^{l+1}\Big]$$

Suppose that problem (1.1),(1.2) has a solution for $\varepsilon = 0$ and some $\varepsilon > 0$. Then, if

$$x'Q(t)x - \varepsilon^{i+1}h_i \geqslant C|x|^2$$

the performance index

$$J_i(u) = J(u) - \varepsilon^{i+1}\int h_i\, dt \tag{4.1}$$

is positive definite with respect to the phase coordinate x. Consequently, the i-th approximation control $u_i(t, x)$ is optimal for system (1.1) with performance index (4.1). By the optimal stabilizing control theorem of /5/,

$$\inf_{u \in U} J_i(u) = W(0, x_0)$$

Consequently,

$$\int |x(t, u_i)|^2 dt \leqslant CW(0, x_0) \tag{4.2}$$

Using the representation (4.1) and condition (4.2) and proceeding as in the case of the zeroth approximation, one can estimate the error in the performance index for higher approximations:

$$V(t, x) = J(u_*) \leqslant J(u_i) \leqslant J_i(u_i) + |J(u_i) - J_i(u_i)|$$

Using (4.1), we have

$$|J(u_i) - J_i(u_i)| \leqslant \varepsilon^{i+1} C \int |x|^2 dt \leqslant \varepsilon^{i+1} C_1 W(0, x_0) \leqslant \varepsilon^{i+1} C_2$$

Thus, the upper and lower estimates are as follows:

$$J(u_*) \leqslant J_i(u_i) + \varepsilon^{i+1} C_2, \; J_i(u_i) \leqslant J(u_*) + |J_i(u_*) - J(u_*)| \leqslant J(u_*) + \varepsilon^{i+1} CW(0, x_0)$$

Consequently, the stabilizing control in the i-th approximation, $u_i(t, x)$ implies an error of the order of ε^{i+1} in the performance index.

5. ***Example***. We consider a model which describes a chain of fermentation conversions of a substrate. The substrate, percolating into a cell, is included in some kind of conversion chain, as a result of which there is an additional biomass exchange. Subject to certain assumptions, the whole growth process can be represented by the scheme illustrated in the figure /7/. Here x_2 is the concentration of the substrate, S_0 is the initial concentration of the substrate (it is assumed that the substrate is supplied at constant concentration), u is the supply rate of substrate to the reactor, E is the concentration of free key enzyme, x_3 is the biopolymer concentration, δ is the stoichiometric coefficient, x_1 is the concentration of the enzyme-substrate complex, K_1 and K_3 are the constants of formation and breakdown rates of the enzyme-substrate complex, K_2 is the formation rate constant of the reaction product. In addition, it is known that E, x_2 and x_1 are related: $\langle E\rangle + \langle x_1\rangle = \varepsilon_1 \langle x_2\rangle$, where the angular brackets denote molar concentration, and ε_1 is the fraction of key enzyme in the overall cell mass $(\varepsilon_1 \ll 1)$.

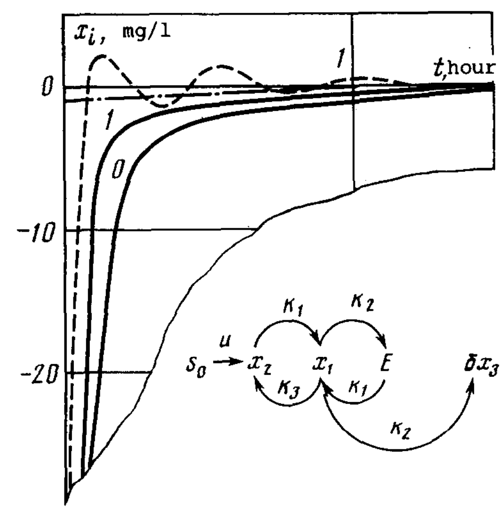

Fig.1

It is assumed that the rate of consumption of the substrate is fairly high and that only a small portion is broken down. The number ε will characterize the breakdown rate. If we put $\varepsilon_1 \approx \varepsilon$ then, using known results /8/, we obtain the following sytem of equations for the dynamics of relative concentrations (the difference between the actual and admissible concentration):

$$x_1^{\cdot} = (-K_1 - \delta K_2 - \varepsilon K_3)\, x_1 + (1 + \varepsilon)\, K_1 x_2 + \varepsilon K_1 x_3 \tag{5.1}$$
$$x_2^{\cdot} = -K_1 x_1 + \varepsilon K_1 x_2 + \varepsilon K_3 x_3 u + (\varepsilon K_3 x_3^* + S_0)\, u$$
$$x_3^{\cdot} = \delta K_2 x_1 - K_d x_3$$
$$x_1(0) = -1, \quad x_2(0) = -60, \quad x_3(0) = -38.06$$

Here x_1 is the (actual) concentration of the enzyme-substrate complex minus the admissible concentration x_1^*. Similarly, x_2 is the substrate concentration minus x_2^* and x_3 is the biopolymer concentration minus x_3^*. The values of the admissible concentrations are: $x_1^* = 1,\ x_2^* = 100,\ x_3^* = 38.06$; $\delta = 66.6$ is the stoichiometric coefficient. The oxidation coefficient is $K_d = 10^{-2}$. The other coefficients are: $K_1 = 0.1,\ K_2 = 0.2,\ K_3 = 0.35,\ \varepsilon = 10^{-2}$. The performance index characterizes the degree to which the concentrations depart from their admissible values:

$$J(u) = \int (x_1^2 + x_2^2 + x_3^2 + u^2)\, dt \tag{5.2}$$

The zeroth- and first-approximation controls are given by the following expressions:

$$u_0(x) = -2758 x_1 - 105{,}5 x_2 - 2095 x_3$$
$$u_1(x) = -0{,}012 x_2^2 - 2758 x_1 - 1915 x_2 - 2095 x_3$$

The figure illustrates the dynamics of the relative concentrations of the components in system (5.1) when the controls $u_0(x)$ (curve 0) and $u_1(x)$ (curve *1*) are applied. The solid curve represents the relative biopolymer concentration, the dashed line represents the relative substrate concentration, and the dash-dot line represents the relative concentration of the enzyme-substrate complex.

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